Stochastic Process Mid Term 2016

Time: Three hours Maximum Marks: 40

Answer any 4 questions. Each question carries 10 marks.

S will denote a countable set (finite or infinite). $\{X_n\}$ will denote an S-valued Markov Chain with stationary transition probabilities (p_{ij}). Let $p_{ij}^{(n)}$ denote the $(i,j)^{th}$ entry in the n^{th} power P^n of the transition probability *matrix* $P = (p_{ij})$. By convention P^0 is taken to be the identity matrix. for $i, j \in S$ and $n \geq 1$, $f_{ij}^{(n)} = P_i(X_n = j,\ X_t \neq j,\ 1 \leq t < n)$ and $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = P_i(X_n = j,\ \text{for some } n \geq 1)$. The period $d(i)$ is defined as

$$d(i) = \gcd\{n \geq 1 : p_{ii}^{[n]} > 0\}.$$

You may use that $i \in S$ is recurrent $\iff f_{ii} = 1 \iff P_i(X_n = i \text{ infinitely often}) = 1 \iff \sum_m p_{ii}^{(m)} = \infty$.

For $i, j \in S$ we say that i leads to j, written as $i \to j$ if $\exists m \geq 0$ such that $p_{ij}^{(m)} > 0$. Further, if $i \to j$ and $j \to i$ then we say that i communicates with j and write it as $i \leftrightarrow j$.

You may use results proved in class but you need to state them precisely and show that conditions are satisfied.

1. For $i, j \in S$ show that

   (a) Show that
   $$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}.$$

   (b) Show that
   $$\sum_{n=1}^{m} p_{ij}^{(n)} \leq f_{ij} \sum_{n=0}^{m} p_{jj}^{(n)}$$

   (c) Show that $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$ if j is transeint.

   (d) Show that a finite state Markov Chain has at least one recurrent state.

2. For $i, j \in S$ and $m \geq 1$ show that

$$P_i(X_n = j \text{ for at least } m \text{ distinct integers } n) = f_{ij}(f_{jj})^{m-1}.$$

3. For $i, j \in S$ such that $i \leftrightarrow j$ show that

(i) Show that $\exists m \geq 1$, $i_1, i_2, \ldots, i_{m-1} \in (S - \{i, j\})$ such that

$$P_i(X_1 = i_1, \ldots X_{m-1} = i_m - 1, X_m = j) = p_{ii_1} p_{i_1 i_2} \ldots p_{i_{m-1} j} > 0.$$

(ii) Show that if i is recurrent then so is j.

(iii) Show that $d(i) = d(j)$.

(iv) Supppose that the Chain admits a stationary initial distribution $\pi = \{\pi_j\}$ and $d(i) = d(j) = 1$. Show that $\pi_i > 0$ if and only if $\pi_j > 0$.

4. Suppose Y_n is a sequence of i.i.d. r.v. with $P(Y_n = r) = c\theta^r$ for $r \geq 1$ where $c = (1 - \theta)/\theta$ and suppose $\{Y_n\}$ is independent of the Markov chain $\{X_n\}$. Let $Z_n = \sum_{k=1}^{n} Y_k$ and let

$$W_n = X_{Z_n}.$$

Show that $\{W_n\}$ is a Markov chain with stationary transition probabilities. Obtain the transition probability matrix of $\{W_n\}$ in perms of $((p_{ij}))$ and θ.

5. **(a)** Suppose $\{X_n\}$ is a Markov chain with state space S and let $Y_n = (X_n, X_{n+1})$. Let T denote the pairs (i, j) such that $p_{ij} > 0$ and show that $\{Y_n\}$ is a Markov chain in the state space T. Write down its transition probabilites. Show that if $\{X_n\}$ is irreducible, aperiodic then so is $\{Y_n\}$. Show that if (π_i) is a stationary distribution for $\{X_n\}$ then $(\pi_i p_{ij})$ is a stationary distribution for $\{Y_n\}$.

(b) Let $S = \{0, 1, 2 \ldots\}$ and let $p_{i,0} = 1 - p$ and $p_{i,i+1} = p$ for all $i \in S$ where $0 < p < 1$. Show that the transition matrix is irreducible and aperidic. Obtain expression for $f_{00}^{[n]}$ and using that show that the chain is recurrent. For what values of p does the chain admit stationary initial distribution? Obatain the same when it exists.